(F-)jumping numbers can be irrational

BGS seminar of Prof. Moraga @ UCLA

X integral / cha 0 or p.

$F^e : X \longrightarrow X$, finite, by Grothendieck duality

$Tr^e : F^e_* \omega_X \longrightarrow \omega_X$

$(X, \Delta, \mathfrak{a}^t)$ $\qquad K_X + \Delta$ $\mathbb{Q}$-Cartier

$\mathfrak{a}$ ideal, $t \geq 0$.

Test ideal $\tau(X, \Delta, \mathfrak{a}^t)$ is the unique

smallest ideal J :

$\forall e$, $\forall \phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(\lceil (p^e-1)\Delta \rceil), \mathcal{O}_X)$

$$\phi\left(F^e_*\left(\mathfrak{a}^{\lceil t(p^e-1)\rceil} J\right)\right) \subset J$$

Multiplier ideal :

$\pi : Y \longrightarrow (X, \Delta, \mathfrak{a})$ RoS exists.

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \pi_* \mathcal{O}_Y\left(\lceil K_Y - \pi^*(K_X + \Delta) - tG\right)$$

$$\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-G)$$

$$\tau(\omega_X, \Gamma, \mathfrak{a}^t) = \tau(X, \Gamma - K_X, \mathfrak{a}^t)$$

(F-) Jumping numbers : λ

$\mathcal{J}(X, \mathfrak{a}^\lambda) \subsetneq \mathcal{J}(X, \mathfrak{a}^\mu)$, $0 \le \mu < \lambda$

τ . . — —

Qn. [Blickle, Schwede, Tucker, Zhang, Patakfalvi . . . .]

Can F-jumping be irrational ? X not $\mathbb{Q}$-Gor.

YES

Result: (B, L M, S, S, T. T, Z, A S)

$(X, \Delta, \mathfrak{a}^t)$ F-jumping's are discrete

set of rational numbers.

when X is $\mathbb{Q}$-Gor, $\tau(X, \mathfrak{a}^t)$

has rational jumps.

dF-H: defined multiplier ideals for

$(X, \mathfrak{a})$ without $\mathbb{Q}$-Gor.

dFH:

$$\mathcal{J}_{dFH}(X, \mathfrak{a}^t) = \sum_{\substack{K_X + \Delta \\ \mathbb{Q}\text{-Cartier}}} \mathcal{J}(X, \Delta, \mathfrak{a}^t)$$

Schwede :

$$\tau_b(X, \mathfrak{a}^t) = \sum_{\substack{K_X + \Delta\ \mathbb{Q}\text{-Carti} \\ p \nmid \operatorname{index}(K_X + \Delta)}} \tau(X, \Delta, \mathfrak{a}^t)$$

Conj [MS] BST : $\tau(X_s, \Delta_s, \mathfrak{a}_s^{\lambda}) = \left(\mathcal{J}(X, \Delta, \mathfrak{a}^{\lambda})\right)_s$

$\forall\ \lambda \geq 0$. $\forall\ s \subset S$ Zariski dense S

Main Result :

Given any $g \geq 2$ $\exists\ (R, \mathfrak{m})$ char p
any $p \geq 3$

$\dim R \geq 3$ with all but finitely many

F-jumping # algebraic irrational of deg g.
$n + K$

Qn transcendental ??

All these examples have all jumping

numbers irrational $K + n$, $n \geq 0$.

Jumps are roots of B-polynomial.

"Smooth"

Tool : Schwede - Tucker, 2012

Thm (see ST) :

$(X = \operatorname{Spec} R\ .\ \Delta,\ \mathfrak{a}^t)$ Suppose, $\exists\ \pi : Y \longrightarrow X$

log RoS which is normalized blow up

$\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-G)$

Then s.t. $\mathcal{I}(X, \Delta, \mathfrak{a}^t) = \mathcal{J}(X, \Delta, \mathfrak{a}^t)$

$\exists T_0 > 0, \quad \forall t \geq T_0$

" Generalization to well-known SNC case "
uses Serre vanishing

Example: (Urbinati)

$X = E \times E$ $\qquad \overline{\mathrm{Eff}} = \mathrm{Nef}$, boundary : $D^2 = 0$

$$\overline{NE}(X)_{\mathbb{R}} = \left\{ x f_1 + y f_2 + z \Delta \;\middle|\; \begin{array}{l} x + y + z \geq 0 \\ xy + yz + zx \geq 0 \end{array} \right\}$$

pth : $\inf \{ t \mid tA - K_W \text{ pseudo eff} \}$
$= t_0(W, A)$

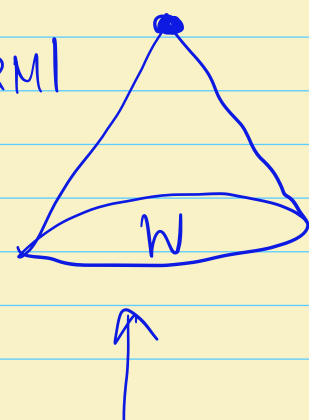
$L_k = 3(2f_1 + f_2 + k\delta)$ $\qquad$ ample

$M = 3(f_1 + f_2)$. $\qquad$ ample.

$2M$ gg.

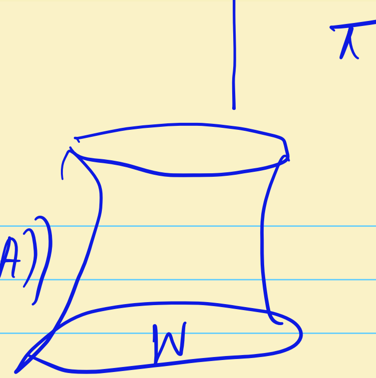
$$t_0 = \frac{2k + 3 + \sqrt{4k^2 + 1}}{2(3k + 1)}, \quad K = 1 - t_0 \notin \mathbb{Q}$$

$$\begin{array}{ll} W & K_W \sim f^* M \\ f \big\downarrow & \text{double-cover.} \\ X & \text{branched along } B \in |2M| \\ & \qquad \text{smooth.} \end{array}$$



C

Not $\mathbb{Q}$-Gor as $K_W \not\sim_{\mathbb{Q}} A$



$$C = \operatorname{Spec}\left(\bigoplus_{m \geq 0} H^0(W, \mathcal{O}_W(mA))\right)$$

$$\mathcal{J}(C, \Gamma, \mathfrak{m}^{n+\kappa}) \subseteq \mathfrak{m}^{n+2}$$

$$\mathcal{J}(C, \Gamma, \mathfrak{m}^{n+\kappa-\varepsilon}) \subset \mathfrak{m}^{n+1}$$

$$\Rightarrow \tau_b(C, \mathfrak{m}^{n+\kappa}) = \mathfrak{m}^{n+2}, \quad \tau(C, \mathfrak{m}^{n+\kappa-\varepsilon}) = \mathfrak{m}^{n+1}, \; n \geq N_0$$

τ or $\mathcal{J}(C, \mathfrak{m}^t)$ has (F-)jumping numbers of the form $n + \kappa$

Qn If λ is F-jump then $\lambda + n$ is F-jump for infinitely many n?

$\lambda \geq \mu(\mathfrak{a})$

$\lambda + 1$

These examples produce Γ_s, $s \in \mathbb{Q}$.

$(C, \Gamma_s, \mathfrak{m})$ having jumps "$1-s$"

$s \downarrow t_0$

$(\mathbb{C}, m)$ jump 1-to

" p if λ is a F-jump then

$p^b \lambda$ is also F-jump "

$\{p^b K\}$

Qn. Transcendental?

There are many other examples (all algebraic) like abelian, K3 surfaces, E^g, $g \geq 2$ etc.

Thank you for listening :)